A Note about the Chain Rule

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I wrote this note in order to clarify to myself some usage of the chain rule I was not familiar with (or had forgotten over the years). I thought this might help others so I wrote it up.

Using the 2-dimensional Chain Rule

CHAIN RULE

$$\frac{\partial}{\partial\cdot} = \frac{\partial x}{\partial\cdot}\frac{\partial}{\partial x} + \frac{\partial t}{\partial\cdot}\frac{\partial}{\partial t} = \frac{\partial}{\partial x}\frac{\partial x}{\partial\cdot} + \frac{\partial}{\partial t}\frac{\partial t}{\partial\cdot}$$

For example:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \qquad \text{where} \quad z = f(x,y) \quad \text{and} \quad x = f(t), y = f(t)$$

Suppose that we have $f(U, \frac{\partial U}{\partial \tau}, \frac{\partial U}{\partial \xi}, \frac{\partial^2 U}{\partial \xi^2})$ where $U = U(\xi, \tau)$ and that we want to change variables to obtain a function of (x, t):

$$U(\xi, \tau) = W(x, t)$$

Establish the change of variables:

$$\begin{aligned} x &= f_1(\xi, \tau) \qquad t = f_2(\xi, \tau) \qquad \text{then} \\ \xi &= g_1(x, t) \qquad \tau = g_2(x, t) \end{aligned}$$

$$dU = dx\frac{\partial U}{\partial x} + dt\frac{\partial U}{\partial t} \quad \rightarrow \quad \frac{\partial U}{\partial\cdot} = \frac{\partial x}{\partial\cdot}\frac{\partial U}{\partial x} + \frac{\partial t}{\partial\cdot}\frac{\partial U}{\partial t} \quad \rightarrow \quad \frac{\partial}{\partial\cdot} = \frac{\partial x}{\partial\cdot}\frac{\partial}{\partial x} + \frac{\partial t}{\partial\cdot}\frac{\partial}{\partial t}$$

Apply this to ξ, τ to get the operators for our change of variables:

$$\frac{\partial}{\partial \xi} = \frac{\partial x}{\partial \xi}\frac{\partial}{\partial x} + \frac{\partial t}{\partial \xi}\frac{\partial}{\partial t} \qquad \text{and} \qquad \frac{\partial}{\partial \tau} = \frac{\partial x}{\partial \tau}\frac{\partial}{\partial x} + \frac{\partial t}{\partial \tau}\frac{\partial}{\partial t}$$

Example (as seen in the Black-Scholes derivation)

$$\frac{\partial U}{\partial \tau} = \frac{1}{2}\sigma^2\frac{\partial^2 U}{\partial \xi^2} + K\frac{\partial U}{\partial \xi}$$

We want to convert (ξ, τ) to (x, t):

$$U(\xi, \tau) = W(x, t)$$

where we wish to apply a translation

$$x = \xi + K\tau \qquad \text{and} \qquad t = \tau$$

in order to get rid of the K term.
We find the partial derivatives for our parameters:

$$\frac{\partial x}{\partial \xi} = 1 \qquad \frac{\partial t}{\partial \xi} = 0 \qquad \frac{\partial x}{\partial \tau} = K \qquad \frac{\partial t}{\partial \tau} = 1$$

Then specialize the chain rule for your parameters:

$$\frac{\partial}{\partial \xi} = \frac{\partial x}{\partial \xi}\frac{\partial}{\partial x} + \frac{\partial t}{\partial \xi}\frac{\partial}{\partial t} = 1\frac{\partial}{\partial x} + 0\frac{\partial}{\partial t} = \frac{\partial}{\partial x} \tag{1}$$

$$\frac{\partial}{\partial \tau} = \frac{\partial x}{\partial \tau}\frac{\partial}{\partial x} + \frac{\partial t}{\partial \tau}\frac{\partial}{\partial t} = K\frac{\partial}{\partial x} + 1\frac{\partial}{\partial t} = K\frac{\partial}{\partial x} + \frac{\partial}{\partial t} \tag{2}$$

And apply the operators to our function U to get the equivalence in terms of W, x and t:

$$\frac{\partial U}{\partial \tau} = K\frac{\partial W}{\partial x} + \frac{\partial W}{\partial t} \qquad \frac{\partial U}{\partial \xi} = \frac{\partial W}{\partial x} \qquad \frac{\partial^2 U}{\partial \xi^2} = \frac{\partial^2 W}{\partial x^2}$$

Replacing into our original equation, one term cancels out and we obtain a Heat equation (with a known solution):

$$\cancel{K\frac{\partial W}{\partial x}} + \frac{\partial W}{\partial t} = \frac{1}{2}\sigma^2\frac{\partial^2 W}{\partial x^2} + \cancel{K\frac{\partial W}{\partial x}} \tag{3}$$

$$\frac{\partial W}{\partial t} = \frac{1}{2}\sigma^2\frac{\partial^2 W}{\partial x^2} \tag{4}$$

Chain Rule with Exponential/Logarithm

Another chain rule trickiness that seems to twist with my mind a bit funny is the usage of an exponential/logarithmic change of variable.

$$S = e^{-\xi} \qquad \frac{\partial}{\partial S} = \frac{\partial \xi}{\partial S}\frac{\partial}{\partial \xi} = \frac{1}{S}\frac{\partial}{\partial \xi} = e^{\xi}\frac{\partial}{\partial \xi} \qquad \frac{\partial}{\partial \xi} = S\frac{\partial}{\partial S} = e^{-\xi}\frac{\partial}{\partial S}$$

Chain Rule with Exponential/Logarithm (II)

This is a great trick to reduce expressions to a total derivative. In the following $f : f(x)$, $g : g(x)$, $h : h(x)$:

$$d(f \cdot g) = df \cdot g + f \cdot dg$$

$$g^{-1} d(f \cdot g) = df + f \cdot \frac{dg}{g}$$

By putting $g(x) = g^{h(x)}$,

$$dg = g \cdot dh = e^{h(x)} dh$$

$$e^{-h(x)} d(e^{h(x)} \cdot f) = df + f \cdot dh$$

e.g.

$$dV_i - dV_a - (V_i - V_a) r dt$$

$$d(V_i - V_a) + (V_i - V_a)(-r dt)$$

$$e^{rt} d(e^{-rt}(V_i - V_a))$$