Stochastic Calculus Cheatsheet

Standard Brownian Motion / Wiener process

$$E[dX] = 0 \qquad E[dX^2] = dt$$
$$\lim_{dt\to 0} dX^2 = dt$$
Discrete approx: $dX = \phi\sqrt{dt}$ where $\phi \sim N(0,1)$

dX is $O(dt^{1/2})$ $\qquad dtdX$ is $O(dt^{3/2})$

Characterization:

1. $X(0) = 0$
2. Continuous everywhere, differentiable nowhere
3. $X(t) - X(s) \sim N(0, |t-s|)$
4. $X(t+s) - X(t)$ is independent of $X(t)$

Levy's characterization:

3. X_t is a martingale w.r.t. the filtration $\mathcal{F}_t$
4. $|X|^2 - t$ is a martingale w.r.t. the filtration $\mathcal{F}_t$

Itô Product Rule

If $dX_t = \alpha dt + \beta dW_t$ and $dY_t = \gamma dt + \lambda dW_t$,

$$\begin{aligned} d(X_tY_t) &= X_t dY_t + Y_t dX_t + dX\,dY \\ &= X_t dY_t + Y_t dX_t + \frac{1}{2}\beta\lambda dt \end{aligned}$$

Stochastic Differential Equations (General Form)

$$\begin{aligned} dS &= f(t,S)\,dt + g(t,S)\,dX_i \\ dS_i &= f_i(t,S_0,\ldots,S_n)\,dt + g_i(t,S_0,\ldots,S_n)\,dX_i \qquad \text{where} \quad f \text{ is the drift, } g \text{ is the diffusion} \end{aligned}$$

Itô's Lemma and Basic Stochastic Integration

For $F(X_t)$

$$dF = \frac{dF}{dX}dX_t + \frac{1}{2}\frac{d^2F}{dX^2}dt \qquad F(X_t) = F(X_0) + \int_0^t \frac{dF}{dX}dX_\tau + \frac{1}{2}\int_0^t \frac{d^2F}{dX^2}d\tau$$

For $F(X_t, t)$

$$dF = \frac{\partial F}{\partial X}dX_t + \left(\frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}\right)dt \qquad F(X_t,t) = F(X_0,0) + \int_0^t \frac{\partial F}{\partial X}dX_\tau + \int_0^t \left(\frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}\right)d\tau$$

Functions of Stochastic Functions

1-dimensional: $V(t,S)$

$$\begin{aligned} dV &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}g^2\frac{\partial^2 V}{\partial S^2}dt \\ &= \left(\frac{\partial V}{\partial t} + f\frac{\partial V}{\partial S} + \frac{1}{2}g^2\frac{\partial^2 V}{\partial S^2}\right)dt + g\frac{\partial V}{\partial S}dX \end{aligned}$$

1. Apply Taylor expansion on V
2. Apply Itô's Lemma:
   - $dX_i^2 \to dt$
   - $dX_i dX_j \to \rho_{ij}dt$
3. Regroup the terms in dt and dX_i
4. Sto.integ.: integrate the resulting DE

2-dimensional: $V(t,S_1,S_2)$

$$dV = \left(\frac{\partial V}{\partial t} + f_1\frac{\partial V}{\partial S_1} + f_2\frac{\partial V}{\partial S_2} + \frac{1}{2}g_1^2\frac{\partial^2 V}{\partial S_1^2} + \rho g_1 g_2\frac{\partial^2 V}{\partial S_1\partial S_2} + \frac{1}{2}g_2^2\frac{\partial^2 V}{\partial S_2^2}\right)dt + g_1\frac{\partial V}{\partial S_1}dX_1 + g_2\frac{\partial V}{\partial S_2}dX_2$$

n-dimensional: $V(t,S_1,\ldots,S_n)$

$$dV = \left(\frac{\partial V}{\partial t} + \sum_{i=1}^n f_i\frac{\partial V}{\partial S_i} + \frac{1}{2}\sum_{i=1}^n g_i^2\frac{\partial^2 V}{\partial S_i^2} + \sum_{i=1,j>1}^n \rho_{ij}\,g_i\,g_j\frac{\partial^2 V}{\partial S_i\partial S_j}\right)dt + \sum_{i=1}^n g_i\frac{\partial V}{\partial S_i}dX_i$$

Transition Density Functions

Forward Kolmogorov

$$\frac{\partial p}{\partial t'} = \frac{1}{2}\frac{\partial^2}{\partial y'^2}\left(B(y',t')^2 p\right) - \frac{\partial}{\partial y'}\left(A(y',t')p\right)$$

Solution

$$p(S,t;S',t') = \frac{1}{\sigma S'\sqrt{2\pi(t'-t)}} e^{-\frac{\left(\log\frac{S}{S'} + \left(\mu - \frac{1}{2}\sigma^2\right)(t'-t)\right)^2}{2\sigma^2(t'-t)}}$$

Common Processes/Dynamics

Brownian Motion with Drift

$$dS = \mu\, dt + \sigma\, dX$$

Vasiček (1977)

$$dS = \gamma(\bar{r} - r)\, dt + \sigma\, dX$$

FIXME TODO add others, Ho Lee and company...

Geometric Brownian Motion (Lognormal)

$$dS = \mu S\, dt + \sigma S\, dX$$

$$\frac{dS}{S} = \mu\, dt + \sigma\, dX$$

Cox, Ingersoll, Ross

$$dS = (\upsilon - \sigma S)\, dt + \sigma S^{\frac{1}{2}}\, dX$$

All you need to know about Sto.Calc

(FIXME integrate these words of wisdom from Antoine.)

- If $X_t \to N(\mu, \sigma)$ then $E(x^{X_t}) = e^{\mu + \frac{\sigma^2}{2}}$.
- Itô: $d(f(X_t))$
- Itô: $d(X_t Y_t) = X_t dY_t + Y_t dX_t + \frac{1}{2}\beta\lambda dt$ where $dX_t = \alpha dt + \beta dW_t$ and $dY_t = \gamma dt + \lambda dW_t$
- $E[\int X_t dW_t] = 0$
- $Var[\int X_t dW_t] = \int X_t^2 dt$
- Girsanov's theorem.
- Generating correlated X and Y.

Martingales

Probability Spaces

"Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space…"

- Ω: sample space
- $\mathcal{F}$: filtration (information set), (Note that $\mathcal{F}_{t_1} \subseteq \mathcal{F}_{t_2} \subseteq \mathcal{F}_T \equiv \mathcal{F}$)
- $\mathbb{P}$: probability measure

Martingales (Definition)

$$\begin{aligned} &\boldsymbol{E}[M_t] < \infty \\ &\boldsymbol{E}[M_{t+1}|\mathcal{F}_t] = M_t \quad \forall 0 \le s \le t \\ &\boldsymbol{E}[M_{t+1}|\mathcal{F}_t] \le M_t \quad \text{(supermartingale)} \\ &\boldsymbol{E}[M_{t+1}|\mathcal{F}_t] \ge M_t \quad \text{(submartingale)} \end{aligned}$$

Wiener $\in$ Martingale (driftless) $\subset$ Markov (memoryless) $\subset$ non-Markov

Equivalent Measures

Absolute continuity: if $P(A) = 0 \to Q(A) = 0 \qquad \forall A$.
$\mathbb{Q}$ is "absolutely continuous" w.r.t. $\mathbb{P}$, and $\mathbb{Q} << \mathbb{P}$.

"It is allright to tinker with the probabilities as long as we do not tinker with the (im)possibilities."

Equivalent measures: if $\mathbb{Q} << \mathbb{P}$ and $\mathbb{P} << \mathbb{Q}$.

Radon-Nikodým Theorem

$\mathbb{Q}(A) = \int_A \Lambda \, d\,\mathbb{P}$ where $\Lambda = \frac{d\,\mathbb{Q}}{d\,\mathbb{P}}$ is the R.N. derivative.

Unconditional Expectation

Expected value under a prob. measure (Lebesgue integral):

$$\boldsymbol{E}[h(X)] = \int_\Omega h(x)p(x)dx = \int_\Omega h(x)d(\mathbb{P}(x)) = \int_\Omega h(x)d\,\mathbb{P}$$

$$\boldsymbol{E}[\mathbb{1}_{\{X\in A\}}] = \int_\Omega \mathbb{1}_{\{X\in A\}} d\,\mathbb{P} = \int_A d\,\mathbb{P} = \mathbb{P}(A)$$

Conditional Expectation

(Use these to prove that a process is a Martingale; use the definition.)

1. Linearity: $\boldsymbol{E}[aX + bY|\mathcal{F}] = a\boldsymbol{E}[X|\mathcal{F}] + b\boldsymbol{E}[Y|\mathcal{F}]$
2. **Tower Property**: if $\mathcal{F} \subset \mathcal{G}$,
$$\boldsymbol{E}[\boldsymbol{E}[X|\mathcal{G}]\,|\mathcal{F}] = E[X|\mathcal{F}]$$
$$\boldsymbol{E}[\boldsymbol{E}[X|\mathcal{F}]] = E[X]$$
3. Taking out what is known:
$$\boldsymbol{E}[X|\mathcal{F}] = X$$
(if X is $\mathcal{F}$-measurable but not Y:) $\quad \boldsymbol{E}[XY|\mathcal{F}] = X\boldsymbol{E}[Y|\mathcal{F}]$
4. Independence: if X is independent from $\mathcal{F}$, $\boldsymbol{E}[X|\mathcal{F}] = E[X]$
5. Positivity: if $X \ge 0$ then $\boldsymbol{E}[X|\mathcal{F}] \ge 0$
6. Jensen' Inequality: if f is a convex function, then $f(\boldsymbol{E}[X|\mathcal{F}]) \le \boldsymbol{E}[f(X)|\mathcal{F}]$

Exponential Martingale

$M(t) = \exp(S_t + f(t))$ where $f(t) = -(\mu + \frac{1}{2}\sigma^2)t$

Itô Integrals & Martingales

Itô integrals are Martingales:

$$\boldsymbol{E}[\int_0^T g(t, X_t)dX_t] = 0$$

Martingale Representation Theorem

If M is a Martingale, there exists $g(t, X)$ such that

$$M_T = M_0 + \int_0^T g(t, X)dX_t$$

The rightmost term is an Itô integral (and thus also a Martingale).

Fubini's Theorem

$$\boldsymbol{E}\left[\int_0^T f(X_t)dt\right] = \int_0^T \boldsymbol{E}\left[f(X_t)\right] dt$$

Properties of Itô Integrals

1. Linearity:
$$\int_0^T (\alpha f(t) + \beta g(t))dX_t = \int_0^T \alpha f(t)dX_t + \int_0^T \beta g(t)dX_t$$
2. Isometry:
$$\boldsymbol{E}\left[\left|\int_0^T f(t)dX_t\right|^2\right] = \boldsymbol{E}\left[\int_0^T |f(t)|^2 dt\right]$$
3. Martingale:
$$\boldsymbol{E}\left[\int_0^T f(t)dX_t \middle| \mathcal{F}_s\right] = \int_0^s f(t)dX_t$$

Application of Martingales to Asset Pricing

Warning: I still need to complete and arrange this page of notes.

Fundamental Asset Pricing Formula

$$\text{Value} = \boldsymbol{E}^{\text{Meas.}}\left[PV(\text{expected cash flows})\right]$$

Risk-free Asset

$$dB_t = rB_t dt, \quad B(0) = B_0$$
$$B(t) = B_0 e^{rt}$$

Underlying S

$$dS_t = \mu S_t dt + \sigma S_t dX, \quad S(0) = S_0$$
$$S(t) = S_0 e^{\mu t - \frac{1}{2}\sigma^2 + \sigma X_t}$$

Removing the TVM

$$S^*(T) = \frac{S(T)}{e^{rt}}$$
$$S^*(t) = S_0^* e^{(\mu - r - \frac{1}{2}\sigma^2)t + \sigma X_t}$$
$$dS^* = (\mu - r)S^* dt + \sigma S^* dX$$

Self-financing Portfolios

Trading Strategy:

$$\phi_t = (\phi_t^S, \phi_t^B) \quad \text{processes}$$

Self-financing portfolio: no in/out flows.

$$\text{Value}: \quad V_t(\phi) = \phi_t^S S_t + \phi_t^B B_t \qquad \forall t \in [0, T]$$
$$V_t(\phi) = V_0(\phi) + \int_0^t \phi_u^S dS_u + \int_0^t \phi_u^B dB_u$$

Arbitrage opportunity:

$$V_0(\phi) = 0$$
$$\text{with } P(V_T(\phi) > 0) > 0 \quad \text{and} \quad P(V_T(\phi) < 0) = 0$$

Novikov Condition

$$E\left[e^{\frac{1}{2}\int_0^T \theta_s^2 ds}\right] < \infty$$

$$M_t^\theta = e^{(-\int_0^t \theta_s dXs - \frac{1}{2}\int_0^t \theta_s^2 ds)} \quad \text{is a Martingale}$$

Girsanov's Theorem

$$\frac{d\,\mathbb{Q}}{d\,\mathbb{P}} = e^{(-\int_0^t \theta_s dXs - \frac{1}{2}\int_0^t \theta_s^2 ds)}$$
$$X_t^{\mathbb{Q}} = X_t^{\mathbb{P}} + \int_0^t \theta(s) ds$$

- Provides an expression for the Radon-Nikodým derivative.
- Gives an explicit correspondence btw $\mathbb{P}$ and $\mathbb{Q}$ in terms of their Brownian motion.

…but does **not** tell you what θ is. We assume θ and check that it satisfies the Novikov condition. Then we have the RN derivative, and we can change measures!

Doléans/Stochastic Exponential

$$\mathcal{E}\left(\int_0^t \theta_s dX_s\right) = \exp\left(\int_0^t \theta_s dX_s - \frac{1}{2}\int_0^t \theta_s^2 ds\right)$$
$$X_t^{\mathbb{Q}} = X_t^{\mathbb{P}} - \int_0^t \theta(s) ds$$

Feynman-Kač Equivalence

$$\text{PDE:} \quad \frac{\partial V}{\partial t} + \mu\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial S^2} - rV = 0, \quad V(T, S) = G(S)$$
$$dS_t = \mu(t, S_t)dt + \sigma(t, S_t)dX_t$$

$$\Updownarrow$$

$$\text{Expectation:} \quad V(t, S_t) = e^{-r(T-t)}\boldsymbol{E}\left[G(S_T)|\mathcal{F}_t\right]$$